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The Special-Relativistic Velocity-Addition Law and a Related Functional Equation[1]

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An attempt is made to derive the velocity composition laws in the special theory of relativity from first principles, i.e., independent of the Lorentz transformations. This leads to a functional equation whose general solution generates an infinity of velocity addition laws besides the Einsteinian ones.

INTRODUCTION

In the special theory of relativity the velocity-addition laws follow from the transformation laws of space-time, i.e., the Lorentz transformations. It has been suggested by Piaget (1961) that from the epistemological viewpoint the concept of velocity in the special theory of relativity should play a role more fundamental than the notion of space-time, and he mentions an attempt by Abelé and Malvaux (1954) to derive the velocity addition laws independently and thence the Lorentz transformations from the transformation laws of velocities.

In the following we present an attempt to derive the velocity composition laws independently, from first principles. We are led to a functional equation, the so-called transitivity equation, whose general solution involves a single function ϕ of one variable only. The Galilean law of $u \pm v$ belongs to the case: (i) Range $\phi = (-\infty, \infty)$. If one imposes the velocity of light c as a limiting velocity in all inertial frames, we have the case: (ii) Range $\phi = (-c, c)$, to which belongs Einstein law of addition of velocities for $\phi(x) = c \tanh x$. However, there are an *infinity* of functions ϕ which belong to the second case and each one of which gives c as a limiting velocity.

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The following proposition illustrates a basic property of the Einstein velocity-addition laws.

Let $\{S_\alpha\}_{\alpha \in I}$ denote the set of all inertial frames moving along the x axis, say. Here I is a countable index set.

Proposition. Let $f: \{S_\alpha\}_{\alpha \in I} \to \mathbb{R}^+$ be any positive function. Assume that the velocity of S_β relative to S_α is $u_{\beta\alpha} = [f(S_\alpha) - f(S_\beta)]/[f(S_\alpha) + f(S_\beta)]$. Then the Einstein velocity-addition law follows.

Proof. We have

$$(1 + u_{\beta\alpha})/(1 - u_{\beta\alpha}) = [f(S_\beta)]/[f(S_\alpha)]$$
$$= [f(S_\beta)\cdot f(S_\gamma)]/[f(S_\gamma)\cdot f(S_\alpha)] = \frac{1 + u_{\beta\gamma}}{1 - u_{\beta\gamma}}\cdot\frac{1 + u_{\gamma\alpha}}{1 - u_{\gamma\alpha}}$$

Therefore,

$$u_{\beta\gamma} = (u_{\beta\alpha} - u_{\gamma\alpha})/(1 - u_{\beta\alpha}\cdot u_{\gamma\alpha})$$

A FUNCTIONAL EQUATION FOR THE VELOCITY TRANSFORMATION LAW

For simplicity, we shall always consider motion in one dimension only, e.g., along the x axis. The general case can be treated analogously. Consider three inertial frames S, S', S'', with their x axis oriented parallel along the same direction. Let v be the velocity of S' relative to S, $\bar{v}$ the velocity of S'' relative to S' and $\bar{\bar{v}}$, the velocity of S'' relative to S. Consider now a particle with velocity u relative to S, with velocity u' relative to S', and u'' relative to S''. Let f be the function of two variables which gives the transformation law of velocities. That is,

$$u' = f(u, v), \qquad u'' = f(u', \bar{v}) \tag{1}$$

We have also $u'' = f(u, \bar{\bar{v}})$. But the same function f also relates the three inertial frames. Namely,

$$\bar{v} = f(\bar{\bar{v}}, v) \tag{2}$$

Thus from (1) and (2), we have

$$u'' = f(u, \bar{\bar{v}}) = f(f(u, v), f(\bar{\bar{v}}, v)) \tag{3}$$

In other words, f satisfies a functional equation of the form

$$f(x, y) = f(f(x, z), f(y, z)) \tag{I}$$

This is the so-called transitivity equation (Aczél, 1966; Schweitzer, 1911) and its solution is given by the following theorem.

Theorem. The general local solution of the functional equation (I) is of the form

$$f(x, y) = \phi[\phi^{-1}(x) - \phi^{-1}(y)]$$

where ϕ is a continuously differentiable and strictly monotone function of one variable, if the domain of (I) is such that f possesses continuous partial derivatives and $f_1(x, z) \neq 0, f_2(x, z) \neq 0$.

Proof. We differentiate (I) with respect to x, y, and z, respectively, to get

$$f_1(x, y) = f_1(f(x, z), f(y, z))f_1(x, z) \tag{4}$$

$$f_2(x, y) = f_2(f(x, z), f(y, z))f_1(y, z) \tag{5}$$

$$0 = f_1(f(x, z), f(y, z))f_2(x, z) + f_2(f(x, z), f(y, z))f_2(y, z) \tag{6}$$

Thus

$$\frac{f_1(x, y)}{f_2(x, y)} = -\frac{f_1(x, z)/f_2(x, z)}{f_1(y, z)/f_2(y, z)} \tag{7}$$

Let

$$\psi(x) = \int \frac{f_1(x, z)}{f_2(x, z)}\, dx \qquad \text{(keeping } z \text{ fixed)} \tag{8}$$

Then from (7)

$$\frac{f_1(x, y)}{f_2(x, y)} = -\frac{\psi'(x)}{\psi'(y)} \tag{9}$$

which is a sufficient condition that $f(x, y)$ and $\psi(x) - \psi(y)$ are functionally related. Thus there exists a function F, such that

$$f(x, y) = F(\psi(x) - \psi(y)) \tag{10}$$

Substituting (10) in (I) we get

$$F(\psi(x) - \psi(y)) = F\{\psi[F(\psi(x) - \psi(z))] - \psi[F(\psi(y) - \psi(z))]\}$$

Let $\psi(x) = \xi$, $\psi(y) = \eta$, $\psi(z) = \zeta$. Then we have

$$\xi - \eta = \psi[F(\xi - \zeta)] - \psi[F(\eta - \zeta)] \tag{11}$$

If we differentiate (11) first with respect to ξ and then η we obtain

$$\frac{\partial}{\partial \xi}\psi[F(\xi - \zeta)] = \frac{\partial}{\partial \eta}\psi[F(\eta - \zeta)] \tag{12}$$

the left-hand side of which is the same as $\partial\psi[F(\xi - \zeta)]/\partial(\xi - \zeta)$. So differentiating (12) with respect to x now, we have

$$\frac{\partial^2\psi[F(\xi - \zeta)]}{\partial(\xi - \zeta)^2}\frac{d\xi}{dx} = 0$$

If $d\xi/dx = 0$, $\psi(x) = $ constant or $f(x, y) = $ constant, which we exclude. If $d\xi/dx \neq 0$ we get

$$\frac{\partial^2 \psi[F(\xi - \zeta)]}{\partial(\xi - \zeta)^2} = 0 \tag{13}$$

whose general solution is $\psi[F(\xi - \zeta)] = c_1(\xi - \zeta) + c_2$ where c_1, c_2 are constants. Let $\chi(x) = \psi(x) - c_2$. Then $\chi[F(\xi - \zeta)] = c_1(\xi - \zeta)$. Hence

$$\chi[f(x, y)] = c_1[\chi(x) - \chi(y)] \tag{14}$$

Substituting (14) in (I) or in $\chi[f(x, y)] = \chi[f(f(x, z), f(y, z))]$ we get

$$c_1[\chi(x) - \chi(y)] = c_1\{c_1[\chi(x) - \chi(z)] - c_1[\chi(y) - \chi(z)]\}$$

That is, $c_1^2 = c_1$ or $c_1 = 0, 1$. Since $f(x, y) \neq$ constant, we must have $c_1 = 1$. So finally, from (14)

$$f(x, y) = \chi^{-1}[\chi(x) - \chi(y)]$$

or putting $\phi = \chi^{-1}$

$$f(x, y) = \phi[\phi^{-1}(x) - \phi^{-1}(y)]$$

Note that $f_1(x, z) \neq 0$ guarantees from (8) that ψ, and hence χ, are strictly monotone. This completes the proof.

Thus

$$f(u, v) = \phi[\phi^{-1}(u) - \phi^{-1}(v)] \tag{II}$$

From physical considerations we must have $u = f(u, 0)$. That is, $u = \phi[\phi^{-1}(u) - \phi^{-1}(0)]$. Or $\phi^{-1}(0) = 0$. We must also have $-v = f(0, v)$, since the velocity of S relative to S' is $-v$. This implies $\phi^{-1}(-v) = -\phi^{-1}(v)$, that is, ϕ^{-1} is an *odd* function. Therefore, ϕ is also an odd function and f is antisymmetric, i.e., $f(u, v) = -f(v, u)$.

The Galilean velocity-addition law is obtained by setting $\phi(u) = u$ to give $f(u, v) = u - v$. We can classify the functions ϕ into two classes, those with (i) Range $\phi = (-\infty, \infty)$ and those with (ii) *finite* Range ϕ. The Galilean law belongs to the first case. If we now assume that the velocity of light c is the limiting velocity in all inertial frames we obtain the second case.

We assume then, that for all $-c < v < c$,

$$\lim_{u \to \pm c} f(u, v) = \pm c \tag{15}$$

From (II) it follows that, for all $v \in$ Domain ϕ^{-1}

$$\pm c = \phi\left[\lim_{u \to \pm c} \phi^{-1}(u) - \phi^{-1}(v)\right]$$

Since ϕ is monotone, this is possible if and only if $\lim_{u \to \pm c} \phi^{-1}(u) = \pm\infty$, or $\phi(\pm\infty) = \pm c$, i.e., Range $\phi = (-c, c)$. Thus (II) and (15) imply that

ϕ, ϕ^{-1} are odd, monotone functions and that Domain $\phi = (-\infty, \infty)$ but Range $\phi = (-c, c)$.

One such function is of course $\phi(u) = c \tanh u$. Then

$$f(u, v) = c \tanh\left(\tanh^{-1}\frac{u}{c} - \tanh^{-1}\frac{v}{c}\right) = \frac{u - v}{1 - uv/c^2}$$

which is the Einstein addition law. Another such function is $\phi(u) = (2c/\pi)\tan^{-1}(u)$, so that

$$f(u, v) = \frac{2c}{\pi}\tan^{-1}\left(\tan\frac{\pi u}{2c} - \tan\frac{\pi v}{2c}\right) \tag{16}$$

However, one can construct an *infinity* of such functions. Because, if $\phi(x)$ is one such function, then consider $\psi(x) = c \sin[(\pi/2c)\phi(x)]$. Thus $\psi(-x) = -\psi(x)$ and $\lim_{x\to\pm\infty}\psi(x) = \pm c$. *Each such function will give rise to a velocity-addition law satisfying both* (I) *and* (15). It remains an interesting problem to find an extra mathematical (and physically plausible) condition which will single out the Einsteinian function ϕ.

We conclude by considering briefly the general problem of deriving the space-time transformation laws from *arbitrary* velocity addition laws. Let the space-time transformation formulas between S and S' be given by $x' = \phi(x, t)$ and $t' = \psi(x, t)$. Then we have (linearly)

$$u' = \frac{dx'}{dt'} = \frac{\phi_x u + \phi_t}{\psi_x u + \psi_t} = f(u, v) \tag{17}$$

We assume that $f(u, v)$ is analytic in $(-c, c) \times (-c, c)$ and expand it in a power series about $(0, 0)$ to get

$$\begin{aligned}(\phi_x u + \phi_t) = (\psi_x u + \psi_t)[&f(0,0) + uf_1(0,0) + vf_2(0,0)\\ &+ \frac{1}{2!}[u^2 f_{11}(0,0) + 2uvf_{12}(0,0) + v^2 f_{22}(0,0)]\\ &+ \frac{1}{3!}[u^3 f_{111}(0,0) + 3f_{112}(0,0)u^2 v + 3f_{122}(0,0)uv^2\\ &+ v^3 f_{222}(0,0)] + \cdots]\end{aligned} \tag{18}$$

Note that from (I) and (II) we have, in general,

$$f(0, 0) = f_{11}(0, 0) = f_{111}(0, 0) = \cdots = 0 \qquad \text{but} \qquad f_1(0, 0) = 1 \tag{19}$$

Thus

$$\phi_t + \phi_x u = \psi_t\alpha(v) + u[\psi_x\beta(v) + \psi_t\nu(v)] + u^2[\psi_x\delta(v) + \psi_t\eta(v)] + \cdots \tag{20}$$

where

$$
\begin{aligned}
\alpha(v) &= \beta(v) = vf_2(0,0) + \frac{v^2}{2!} f_{22}(0,0) + \frac{v^3}{3!} f_{222}(0,0) + \cdots \\
\nu(v) &= \delta(v) = f_1(0,0) + vf_{12}(0,0) + v^2 f_{122}(0,0) + \cdots \qquad (21) \\
\eta(v) &= vf_{112}(0,0) + \cdots
\end{aligned}
$$

etc.

Equating the coefficients of powers of u in (20) we obtain for the (linear) transformation matrix

$$
A(v) = \begin{pmatrix} \phi_x & \phi_t \\ \psi_x & \psi_t \end{pmatrix} = \begin{pmatrix} \left[\delta(v) - \frac{\eta(v)}{\delta(v)}\right] & \alpha(v)\gamma(v) \\ -\frac{\eta(v)}{\delta(v)} \gamma(v) & \gamma(v) \end{pmatrix} \qquad (22)
$$

where $\psi_t = \gamma(v)$.

For the Lorentz case, $\alpha(v) = -v$, $\delta(v) = 1 + v/c^2$,

$$
\eta(v) = \frac{v}{c^2}\left(1 + \frac{v}{c^2}\right)
$$

and so

$$
A(v) = \begin{pmatrix} \gamma(v) & -v\gamma(v) \\ -\frac{v}{c^2}\gamma(v) & \gamma(v) \end{pmatrix}
$$

One can now use the fact $A(-v)\cdot A(v) =$ identity, to give $\gamma(v)\gamma(-v) = (1 - v^2/c^2)^{-1}$. Finally, spatial isotropy can be invoked to give $\gamma(v) = \gamma(-v) = (1 - v^2/c^2)^{-1/2}$ and thus the Lorentz transformation.

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